Indian Statistical Institute, Bangalore
M. Math Second Year
Second Semester - Ergodic Theory

Semestral Exam — Date: May 04, 2018
Maximum marks: 50 — Duration: 3 hours

Section I: Answer any four, each question carries 6 marks

1. Prove that the two sided $(p_0, p_1, \cdots, p_k)$ shift is ergodic.
2. Find a necessary and sufficient condition on θ_1 and θ_2 for the ergodicity of the map $R(z_1, z_2) = (e^{2\pi i\theta_1} z_1, e^{2\pi i\theta_2} z_2)$ on $\mathbb{T} \times \mathbb{T}$.
3. Consider $T(x) = x^2$ for $x \in \mathbb{T}$. Find the return time of T to the upper semicircle.
4. Let $(X, \mathcal{B}, m)$ be a probability space and $\mathcal{A}, \mathcal{C}, \mathcal{D}$ are finite subalgebras of $\mathcal{A}$. Prove that $H(\mathcal{A} \vee \mathcal{C} \mid \mathcal{D}) = H(\mathcal{A} \mid \mathcal{D}) + H(\mathcal{C} \mid \mathcal{A} \vee \mathcal{D})$ and $H(\mathcal{A} \vee \mathcal{C}) = H(\mathcal{A}) + H(\mathcal{C} \mid \mathcal{A})$.
5. Let $X = [0, 1)$ with Lebesgue measure and $T(x) = x - \frac{2^k-3}{2^k}$ for $x \in (\frac{2^k-2}{2^k}, \frac{2^k-1}{2^k})$ for $k \geq 1$. Prove that T has discrete spectrum.
6. Prove that every Bernoulli automorphism is a Kolmogorov automorphism.

Section II: Answer any two, each question carries 13 marks

1. (a) Let T be a mpt on a probability space $(X, \mathcal{B}, m)$. Prove that T is strong mixing if and only if $\langle U_{T^n} f, f\rangle \to \langle f, 1\rangle\langle 1, f\rangle$ for all $f \in L^2(X)$ (**Marks: 6**).

   (b) Prove that the map $T(z, w) = (ze^{2\pi i\sqrt{2}}, zw)$ is uniquely ergodic on $\mathbb{T}^2$.
2. (a) Compute the entropy of (p, P) Markov shift (**Marks: 6**).

   (b) Prove Gauss map preserves Gauss measure and assuming ergodicity of the Gauss map find the relative frequency of an integer k to appear in the continued fraction almost every $x \in [0, 1]$.
3. (a) Determine when two rotations are spectrally isomorphic (**Marks: 7**).

   (b) Any mpt T with countable Lebesgue spectrum is strong mixing. .